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On the Spectrum of the Rayleigh Piston II

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Building on results of an earlier paper we study the discrete spectrum of the Rayleigh piston. We first prove the absence of discrete spectrum on the subspace of odd functions everywhere in the Lorentz regime. Then we give upper bounds on the number of discrete eigenvectors as a function of the mass ratio using a variety of methods which to some degree complement each other. We also investigate the precise degree of divergence of these bounds as the mass ratio goes to infinity respectively zero.

KEY WORDS:

1. INTRODUCTION

In Ref. 1 we began a study of the spectral properties of the operator family $\{Z - G_\mu\}_{\mu\in(1/2,\infty)}$; $Z(x) := e^{-x^2} + 2x\phi(x)$, $\phi(x) := \int_0^x e^{-t^2}\,dt$, a multiplication operator and G_μ an integral operator with kernel

$$\begin{aligned} g_\mu(x, y) &= \mu^2|x - y|\exp\left[-\tfrac{1}{2}(x^2 + y^2) - \alpha(x - y)^2\right] \\ &= \mu^2|x - y|\exp\left[-2s^2(x^2 + y^2) + \alpha(x + y)^2\right] \qquad (1.1) \end{aligned}$$

with $\alpha = \mu(\mu - 1)$, $s = \mu - \frac{1}{2}$ (these abbreviations will be used throughout); both operators acting on $L^2(\mathbb{R})$. We have $\mu = (2\gamma)^{-1}(1 + \gamma)$ with γ the mass ratio between test- and heat-bath particle in the Rayleigh piston model. For the sake of brevity we refer the reader to the introduction of Ref. 1 for motivation, background literature, and general remarks.

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From Ref. 1 we learn that the discrete spectrum of $Z - G_\mu$ is contained in $[0, 1]$ for all $\mu(e^{-sx^2}$ defining the unique ground state) and empty apart from this ground state near $\mu = \gamma = 1$.

In the present paper we are concerned with the problem of giving upper bounds for the number of discrete eigenvalues (always counting multiplicities) as a function of μ, in particular with the degree of divergence of that bound for $\mu \to \frac{1}{2}$, respectively $\mu \to \infty$. We shall see that the situations for $\mu > 1$ (the Rayleigh regime) and $\mu < 1$ (the Lorentz regime) are radically different in many respects so that a variety of methods will have to be developed to deal with the problem. The strategic problem here is that there are innumerably many ways in which our operators can be transformed and adjusted so as to make applicable one of the general methods which have been developed to solve the same problem for the Schrödinger equation (see, e.g., B. Simon's article in Ref. 2 or the relevant chapters in Ref. 4). Unfortunately the relative merits of each ansatz can only be ascertained after having carried it out. We hope that our choice will also give the reader some insight into the structure of G_μ as well as into the general methods applied to the special case.

At present the question of lower bounds remains open; it is considerably harder since general methods which only need applying do not seem to exist. This can thus only be solved by a much deeper study of the operators involved.

2. ABSENCE OF ODD EIGENFUNCTIONS IN THE LORENTZ REGIME

Before stating the result let us for a moment consider the Rayleigh regime. For $\mu \to \infty$, G_μ converges strongly to the multiplication operator e^{-x^2}; therefore on L^2_{odd} as well as on L^2_{even} the discrete eigenvalues become arbitrarily dense in (1.1) if μ is large enough (see Ref. 1 for details). The heuristics for the statement below is as follows: we first define

$$G_\mu^c := \mu^2 e^{-(1/2+\alpha)(x^2+y^2)} \cosh(2\alpha xy)|x - y| \tag{2.1}$$

and similarly G_μ^s with the sinh function. Then G_μ^c is negative definite on L^2_{odd}, whereas G^s is negative definite on L^2_{even} for $\mu > 1$ but positive definite for $\mu < 1$ (expand the cosh, respectively sinh, term and integrate matrix elements term by term[1]). For $\mu < 1$ we then write

$$Z - G_\mu = Z - 2G_\mu^c + \mu^2 |x - y| \exp\left[-2s^2(x^2 + y^2) - |\alpha|(x - y)^2\right] \tag{2.2}$$

and restricting both sides to L^2_{odd} we have

$$Z - G_\mu\big|_{L^2_{\text{odd}}} \geqslant \left\{Z + \mu^2|x - y| \exp\left[-2s^2(x^2 + y^2) - |\alpha|(x - y)^2\right]\right\}\big|_{L^2_{\text{odd}}}$$

In the limit $\mu \to 1/2$ the right-hand side becomes $Z + \frac{1}{4}|x-y|e^{-(1/4)(x-y)^2}$. This last operator is easily seen to be strictly positive. So the spectrum of $Z - G_\mu$ on L^2_{odd} does not extend down to 0 as is the case on L^2_{even}.[1] Since the estimate above is extremely crude we might hope that $\text{Sp}(Z - G_\mu)$ on L^2_{odd} has lower bound 1 for all $\mu \in (1/2, 1)$. In fact we get the following:

Theorem 1. For all $\mu \in (1/2, 1)$ $\text{Sp}[(Z - G_\mu)_{|L^2_{\text{odd}}}]$ is contained in $[1, \infty)$.

Proof. Let $\mu \in (1/2, 1)$; $(Z - G_\mu)|_{L^2_{\text{odd}}}$ can be rewritten as an operator on $L^2(\mathbb{R}_+)$

$$Z - \mu^2\Big[|x-y|e^{\alpha(x+y)^2} - (x+y)e^{\alpha(x-y)^2}\Big]e^{-2s^2(x^2+y^2)} \tag{2.3}$$

where we have renormalized the state vectors, thus absorbing the factor 2. The operator on $L^2(R_+)$ corresponding to the new kernel becomes unbounded as $\mu \to 1/2$. However, the unbounded part is harmless, as seen above. So our strategy is to find a negative definite kernel representing an operator $-B*B$ that approximates the negative part of $G_{\mu\,|\,L^2_{\text{odd}}}$ better that (2.2) and then estimate the remainder as a form by $\theta(\mu)(Z-1)$ with $\theta(\mu)$ a continuous function on $[\frac{1}{2}, 1]$ and bounded by 1 on the interval. This will mean that for f odd we have $(f, [Z - G_\mu]f) > \|f\|_2^2$ for all $\mu \in (\frac{1}{2}, 1]$ which is what we want.

2. It takes some trial and error to find a suitable kernel for B but eventually one comes up with the ansatz $[(x, y) \in \mathbb{R}^2_+]$:

$$K^{\pm}(x, y) := e^{-ux^2}(e^{-\beta(x-y)^2} \pm e^{-\beta(x+y)^2})e^{-vy^2} \cdot y^{1/2} \tag{2.4}$$

For $x, y \geqslant 0$ we compute:

$$\begin{aligned}
&\int_0^\infty K^-(x, y)K^-(y, z)\,dy \\
&\quad = \exp\Big[-(u+\beta)(x^2+z^2) + \sigma(x+z)^2\Big] \\
&\qquad \times \left\{\int_{-(\sigma/\beta)(x+z)}^{\infty} e^{-\gamma t^2}\Big[t + \frac{\sigma}{\beta}(x+z)\Big]dt \right.\\
&\qquad\quad \left. + \int_{(\sigma/\beta)(x+z)}^{\infty} e^{-\gamma t^2}\Big[t - \frac{\sigma}{\beta}(x+z)\Big]dt\right\} \\
&\quad - \exp\Big[-(u+\beta)(x^2+z^2) + \sigma(x-z)^2\Big] \\
&\qquad \times \left\{\int_{-(\sigma/\beta)(x-z)}^{\infty} e^{-\gamma t^2}\Big[t + \frac{\sigma}{\beta}(x-z)\Big]dt \right.\\
&\qquad\quad \left. + \int_{(\sigma/\beta)(x-z)}^{\infty} e^{-\gamma t^2}\Big[t - \frac{\sigma}{\beta}(x-z)\Big]dt\right\}
\end{aligned}$$

where we have set $\sigma := \beta^2[2(v+\beta)]^{-1}$, $\gamma := 2(v+\beta)$. The terms with integrand te^{-t^2} can be evaluated immediately and cancel out in the case of K^- since $\gamma\sigma^2\beta^{-2} = \sigma$. In the case of K^+ they would contribute $(2/\gamma)$ $\exp[-(u+\beta)(x^2+y^2)]$, about which more will be said below. In the case of K^-, however, we are left with

$$\exp\left[-(u+\beta)(x^2+z^2)\right]\left[\frac{2\sigma}{\beta}e^{\sigma(x+z)^2}(x+z)\int_0^{(\sigma/\beta)(x+z)}e^{-\gamma t^2}\,dt\right.$$
$$\left.-\frac{2\sigma}{\beta}e^{\sigma(x-z)^2}|x-z|\int_0^{(\sigma/\beta)|x-z|}e^{-\gamma t^2}\,dt\right]$$
$$=2\delta\cdot\exp\left[-(u+\beta-2\sigma)(x^2+z^2)\right]$$
$$\times\left\{(x+z)e^{-\sigma(x-z)^2}\int_0^{\sqrt{\sigma}(x+z)}e^{-\tau^2}\,d\tau\right.$$
$$\left.-|x-z|e^{-\sigma(x+z)^2}\int_0^{\sqrt{\sigma}|x-z|}e^{-\tau^2}\,d\tau\right\} \tag{2.5}$$

where again we have abbreviated $\delta := |\sigma|\cdot|\beta|^{-1}\cdot|\gamma|^{-1/2}$. To make this resemble (2.3) we adjust the exponentials setting $\sigma = |\alpha|$, $(u+\beta-2\sigma) = 2s^2$, which yields $u+\beta = 1/2$. The integrals left in (2.5) then give $\phi[|\alpha|^{1/2}(x+z)]$, respectively, $\phi(|\alpha|^{1/2}|x-z|)$, irrespective of the choice of (u, v, β) within the allowed range. Thus this ansatz has its limitations and does not allow to show that G_μ is itself negative definite on the odd subspace. We now fix $u=0$, $\beta = 1/2$, $v = (8|\alpha|)^{-1} - 1/2$. Defining now the operator B^* as having the kernel $\mu\cdot(|\alpha|\pi)^{-1/4}K^-$ we get

$$B^*B = \frac{2\mu^2}{\sqrt{\pi}}e^{-2s^2(x^2+y^2)}\left\{(x+y)e^{\alpha(x-y)^2}\phi\left[|\alpha|^{1/2}(x+y)\right]\right.$$
$$\left.-|x-y|e^{\alpha(x+y)^2}\phi(|\alpha|^{1/2}|x-y|)\right\} \tag{2.6}$$

where we introduced the factor $2/\sqrt{\pi}$ to compensate for the asymptotic value of $\phi(x)$.

In order to further improve the approximation we define another negative definite operator $-A^*A = \mu^2\exp[-(1/2)(x^2+y^2)]|x-y|$, which as a kernel on $L^2(\mathbb{R}_+)$ becomes $\mu^2\exp[-(1/2)(x^2+y^2)][|x-y|-(x+y)]$. On the $L^2(\mathbb{R}_+)$ we then have $Z - G_\mu \geqslant Z - G_\mu - B^*B - A^*A$; and the kernel of $G_\mu + B^*B + A^*A$ becomes

$$\mu^2 e^{-2s^2(x^2+y^2)}\left(|x-y|e^{\alpha(x+y)^2}\left[1-\frac{2}{\sqrt{\pi}}\phi(|\alpha|^{1/2}|x-y|)-e^{\alpha(x-y)^2}\right]\right.$$
$$\left.-(x+y)e^{\alpha(x-y)^2}\left\{1-\frac{2}{\sqrt{\pi}}\phi\left[|\alpha|^{1/2}(x+y)\right]-e^{\alpha(x+y)^2}\right\}\right)$$
$$=\mu^2 e^{-(1/2)(x^2+y^2)}\left\{(x+y)L\big(|\alpha|^{1/2}(x+y)\big)-|x-y|L\big(|\alpha|^{1/2}|x-y|\big)\right\} \tag{2.7}$$

with

$$L(x):=1-\frac{2}{\sqrt{\pi}}e^{x^2}\int_x^\infty e^{-t^2}\,dt$$

What have we achieved so far? The function $L(|\alpha|^{1/2}x)$ is bounded by 1 uniformly for $\mu\in[\frac{1}{2},1]$; so (2.7) represents a "small" Hilbert–Schmidt term down to $\mu=\frac{1}{2}$ unlike the crude decomposition (2.2). In principle we could estimate (2.7) against $Z-1$ numerically but we shall go on to give an analytic estimate for $\theta(\mu)$ first because it is not too much trouble and second because the methods used to estimate the various terms will probably prove useful for handling other problems connected with the operators in question and which may not be accessible numerically.

We first estimate $L(x)$ from above. We are looking for a function $h(x)$ with the following properties:

(a) $h(x)\geqslant L(x)$, $\forall x\geqslant 0$; (b) $h(x)$ and $x[h(x)-L(x)]$ grow monotonically for $x>0$ [note that $L(x)$ itself grows monotonically so that (2.7) is strictly positive in $\mathbb{R}_+$]. With such an $h(x)$ we get

$$\left|(x+y)L\big(|\alpha|^{1/2}(x+y)\big)-|x-y|L\big(|\alpha|^{1/2}|x-y|\big)\right|$$
$$\leqslant\Big|(x+y)h\big(|\alpha|^{1/2}(x+y)\big)-|x-y|h(|\alpha|\,|x-y|)$$
$$+(x+y)\left[h\big(|\alpha|^{1/2}(x+y)\big)-L\big(|\alpha|^{1/2}(x+y)\big)\right]$$
$$+|x-y|\left[h(|\alpha|^{1/2}|x-y|)-L(|\alpha|^{1/2}|x-y|)\right]\Big|$$
$$\leqslant(x+y)h\big(|\alpha|^{1/2}(x+y)\big)-|x-y|h\big(|\alpha|^{1/2}|x-y|\big) \tag{2.8}$$

Noting that near $x=0$ $L(x)\approx 2x/\sqrt{\pi}$ we make a simple ansatz trying to find $a > 0$ such that $L(x) \leqslant 2x[\sqrt{\pi}(1+ax)]^{-1}$ for $x > 0$. This will work if

$$\frac{2x}{\sqrt{\pi}(1+ax)} e^{-x^2} - e^{-x^2} + \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt \geqslant 0 \tag{2.9}$$

The derivative of (2.9) is

$$(2/\sqrt{\pi})\left[-2x^2(1+ax)^{-1} + (1+ax)^{-2} + x - 1\right]e^{-x^2},$$

which on $\mathbb{R}_+$ has the same sign as

$$x(\sqrt{\pi} - 2a) + x^2(2a\sqrt{\pi} - a^2 - 2) + x^3(a^2\sqrt{\pi} - 2a) \tag{2.10}$$

Near zero this has to be > 0 so we must have either $\sqrt{\pi} - 2a > 0$ or $\sqrt{\pi} - 2a = 0$ and $2a\sqrt{\pi} - a^2 - 2 > 0$ (for $a = \sqrt{\pi}/2$ this becomes $3\pi/4 - 2 > 0$). With $a = \sqrt{\pi}/2$ the coefficient of x^3 is < 0; (2.10) has exactly one zero in $(0, \infty)$, i.e., $a = \sqrt{\pi}/2$ is the best possible choice for this type of estimate.

To check condition (b) we need only prove that with our choice $h(x) - L(x)$ grows monotonically. For this we need

$$\int_x^\infty e^{-t^2}\,dt - \frac{1}{2} e^{-x^2} \frac{2a + a^2x}{(1+ax)^2} \geqslant 0, \qquad \forall x > 0 \tag{2.11}$$

This will hold if the derivative of (2.11) is > 0 near $x = 0$ and has exactly one zero in $(0, \infty)$. The possible zeros of that derivative are those of

$$(1+ax)^{-3}\left[-a(1 - a^2/2)x + (3/2)a - 1\right] \tag{2.12}$$

For $a = \sqrt{\pi}/2$ this works.

The function in (2.7) is thus everywhere smaller than

$$\begin{aligned} xyT_\mu(x, y) := {} & \mu^2 e^{-(1/2)(x^2+y^2)} \frac{8|\alpha|^{1/2}}{\sqrt{\pi}} xy \\ & \times \left[1 + \frac{(\pi|\alpha|)^{1/2}}{2}(x+y)\right]^{-1} \left[1 + \frac{(\pi|\alpha|)^{1/2}}{2}|x-y|\right]^{-1} \\ & \times \left[1 + \frac{(\pi|\alpha|)^{1/2}}{2} \frac{(x+y)|x-y|}{(x+y)+|x-y|}\right] \end{aligned} \tag{2.13}$$

and so if both of the corresponding operators are sandwiched between multiplication operators $(Z-1)^{-1/2}$ the norm of the first will be smaller than the norm of the second, which in any case we have to estimate by its Hilbert–Schmidt (HS) norm.

First, using lemma 1 of Ref. 1 we note that the HS norm in question will be smaller than the one of $\exp[(1/12)(x^2+y^2)]T_\mu(x,y)$. Noting further that the new integrand is symmetric with respect to the transformations $(x,y)\to(-x,y)$ and $(x,y)\to(-x,-y)$ we convert the integral over $\mathbb{R}_+^2$ into one over $\mathbb{R}^2$, introduce new variables $u:=(x+y)/2$; $v:=(x-y)/2$, and then go back to $\mathbb{R}_+^2$ to get [setting $q:=(\pi|\alpha|)^{1/2}$]

$$\begin{aligned}\|e^{(1/12)(x^2+y^2)}T_\mu(x,y)\|_{\mathrm{HS}}^2 &= \left[\frac{8\mu^2|\alpha|^{1/2}}{\sqrt{\pi}}\right]^2\cdot 2\int_0^\infty\int_0 e^{-(5/3)(u^2+v^2)}(1+qu)^{-2}\\ &\quad\times(1+qv)^{-2}\left(1+q\frac{uv}{u+v}\right)^2 du\,dv\end{aligned}\tag{2.14}$$

Next we write out $uv=\frac{1}{4}(u+v)^2-\frac{1}{4}(u-v)^2$, multiply out the last bracket in (2.14), neglect the negative contributions, and we see that the integral in (2.14) is smaller than

$$q^{-2}\left[\int_0^\infty e^{-\sigma z^2}\frac{dz}{(1+z)^2}+\frac{1}{2}\int_0^\infty e^{-\sigma z^2}\frac{z\,dz}{(1+z)^2}\right]^2\tag{2.15}$$

having set $\sigma:=5(3\pi|\alpha|)^{-1}$. Thus we get

$$\|e^{(1/12)(x^2+y^2)}T_\mu(x,y)\|_{\mathrm{HS}}\leqslant\mu^2\frac{8\sqrt{2}}{\pi}\int_0^\infty\left(1+\frac{z}{2}\right)e^{-\sigma z^2}\frac{dz}{(1+z)^2}\tag{2.16}$$

The last integral which apparently cannot be computed exactly must be estimated. First, by estimating

$$I(\sigma)'=-\frac{1}{2\sigma}\int_0^\infty e^{-\sigma z^2}\frac{dz}{(1+z)^3}\geqslant-\frac{\sqrt{\pi}}{4}\sigma^{-3/2}$$

we get $I(\sigma)\leqslant\frac{1}{2}(\pi/\sigma)^{1/2}$. This yields a bound $4(1.2)^{1/2}\mu^2|\alpha|^{1/2}$ which is not quite sufficient (taking values >1 near $\mu=5/6$). However, one easily finds

$$I(\sigma)=\frac{1}{4}\left[\left(\frac{\pi}{\sigma}\right)^{1/2}-\frac{1}{\sigma}\right]+\frac{2\sigma+1}{4\sigma}\int_0^\infty e^{-\sigma x^2}\frac{dx}{(1+x)^2}\tag{2.17}$$

From this we get $I(\sigma)\leqslant(2\sigma-1)^{-1}[(\pi\sigma)^{1/2}-1]$. This gives an additional factor to the first bound which has the form $(2\sigma-1)^{-1}[2\sigma-2(\sigma/\pi)^{1/2}]$. It is then straightforward to check that in the critical range of the first bound this factor (which grows monotonically with μ in that range) is small enough to make the new bound smaller that 1 for all $\mu\in[\frac{1}{2},1)$. The theorem is proved.

Remark. From the theorem we derive the existence of three threshold values for the discrete spectrum of $Z - G_\mu$. On L^2_{even} there are μ_+, μ_- with $\mu_- < 1 < \mu_+$ for the onset of the second discrete eigenvalues beside the ground state as one goes away from $\mu = 1$. On L^2_{odd} there is μ_0, the smallest number for which there is a discrete eigenvalue on L^2_{odd}. The methods and results discussed below allow to estimate these values, but we shall not do so since we feel that these estimates would still be rather crude and we hope to find better methods for doing so in the future.

3. A SCHRÖDINGER-TYPE UPPER BOUND IN THE RAYLEIGH REGIME

In this section we give an upper bound for the total number of discrete eigenvalues valid for all $\mu > 1$. Its method consists in reducing the problem to one of estimating the number of discrete eigenvalues of a one-dimensional Schrödinger operator for which a bound in terms of the potential is available.(2) It is somewhat similar to the method used in Ref. 3 to prove finiteness of the number of such eigenvalues but more direct and easier controlled numerically.

As a first step we note that for $\mu > 1$ G_μ is convolution by $\varphi_\alpha(x) := |x|e^{-\alpha x^2}$ sandwiched between multiplication operators. Splitting the Fourier transform $\tilde{\varphi}_\alpha(k)$ into positive and negative parts we get a decomposition of G_μ as a difference of two positive operators (this has nothing to do with the canonical one resulting from spectral theory). Let

$$H_\mu^\pm := \mu^2 e^{-x^2/2} \tilde{\varphi}_\alpha^\pm (k) e^{-y^2/2} \tag{3.1}$$

it then follows that as forms on $L^2(\mathbb{R})$

$$Z - G_\mu \geqslant Z - H_\mu^+ \tag{3.2}$$

and by the min–max principle (Ref. 4, Chapter XIII) we know that $Z - H_\mu^+$—which has the same continuum threshold as $Z - G_\mu$—has at least as many eigenvalues in $(-\infty, 1]$ as has $Z - G_\mu$ in $[0, 1]$.

More precisely this remains true if the interval $(-\infty, 1]$ is replaced by $(-\infty, 1)$ for $Z - H_\mu^+$ (this bit of extra precision will be needed below). For let f be a discrete eigenvector of $Z - G_\mu$. The form of the kernel of G_μ then implies that there is a constant M_f such that for $x \in \mathbb{R} |f(x)| \leqslant M_f e^{-x^2/2}$. Thus the Fourier transform $\tilde{f}$ has an entire continuation into $\mathbb{C}$ and cannot be zero on any real nonzero interval. Thus $(f, H_\mu^- f) > 0$ and, if E denotes for the moment the spectral projection for $[0, 1]$ of $Z - G_\mu$, we get on $EL^2(\mathbb{R})$

$$E(z - H_\mu^+)E < E(Z - G_\mu)E \tag{3.3}$$

so $E(Z - H_\mu^+)E$ has at least as many eigenvectors as $Z - G_\mu$ on $EL^2(\mathbb{R})$. But then by Ref. 4, Theorem XIII.3, $Z - H_\mu^+$ has at least as many eigenvalues in $(-\infty, 0)$ as $Z - G_\mu$ in $[0, 1]$.

Now in Ref. 1 we proved that $Z(x) - 1 \geqslant x^2 e^{-x^2/2}$ for all $x \in \mathbb{R}$; so we can write (shifting the continuum threshold)

$$Z - 1 - H_\mu^+ \geqslant x^2 e^{-x^2} - e^{-x^2/2} \varphi_\alpha^+ (x - y) e^{-y^2/2} =: T_\mu \tag{3.4}$$

Therefore T_μ^-—the strictly negative part of T_μ—has rank not smaller than rank $(Z - 1 - H^+)^-$; however, it is clear that

$$\dim \operatorname{ran}(T_\mu^-) \leqslant \dim \operatorname{ran}\left[x^2 - \tilde{\varphi}_\alpha^+(k)\right]^- \tag{3.5}$$

where "ran" denotes the range of the operator.

The problem is thus reduced to estimating the number of discrete eigenvalues of the one-dimensional Schrödinger operator in (3.5). Now it was also shown in Ref. 1 that $\tilde{\varphi}_\alpha(k)$ goes like $O(k^{-2})$ for $|k|$ large; so it is not immediately obvious that application of Theorem 6.4 of Ref. 2 will give a finite upper bound. The necessary facts about $\tilde{\varphi}_1^+(k)$—of which all the $\tilde{\varphi}_\alpha^+(k)$ are scale transforms—are gathered in the following:

Lemma 2.1. (a) $\tilde{\varphi}_1^+(k)$ is positive in a neighborhood of $k = 0$ and its support is a symmetric interval contained in $(-2, 2)$. (b) On its range we have

$$\cos(k) \leqslant \tilde{\varphi}_1^+(k) \leqslant 1 - k^2/2 + k^4/12 \tag{3.6}$$

Proof. (a) We write

$$\tilde{\varphi}_1(k) = 2\int_0^\infty x e^{-x^2} \cos(kx)\, dx = 2\frac{d}{dk}\left[\int_0^\infty e^{-x^2} \sin(kx)\, dx\right] =: 2\left[\theta(k)\right]_k \tag{3.7}$$

Since e^{-x^2} decreases monotonically on $(0, \infty)$ we have $\theta(k) > 0$, $\forall k > 0$ and θ obeys the following DE:

$$\theta''(k) + k\theta'(k) + \theta(k) = 0 \tag{3.8}$$

If now θ had a local minimum at any point $k_1 > 0$ we would have $\theta''(k_1) < 0$ by (3.8), a contradiction. Thus in $\{k > 0\}\theta(k)$ has just one maximum and so $\theta'(k)$ has exactly one zero κ_1 in $(0, \infty)$. By means of the power series expansion $\tilde{\varphi}_1(k) = \sum_{n=0}^\infty (-1)^n [n!/(2n)!] k^{2n}$ one then readily checks that $\theta'(2) < 0$ and also the double inequality (b). The remaining assertions are trivial. ■

Let us introduce the notation $n_\mu(\lambda)$ for the number of eigenvalues of $Z - G_\mu$ in $[0, \lambda]$ with the additional subscript e, respectively o, if only the even, respectively, odd, subspace are considered. From the formula quoted above we then have the bound

$$n_\mu(1) \leqslant 1 + \mu^2 \int_0^{\kappa\mu} k\tilde{\varphi}_\alpha^+(k)\,dk = 1 + \mu^2 \int_0^{\kappa_1} k\tilde{\varphi}_1^+(k)\,dk \tag{3.9}$$

Using part (b) of the lemma above we see that the μ dependence of our bound is not better than $(\pi/2 - 1)\mu^2$, that is, the bound is weak near $\mu = 1$, a feature it shares with all the other global bounds below (for obvious reasons). Though we might try to improve the overall constant in the bound by considering the even and odd subspaces separately this would not change the μ dependence, i.e., if there were a marked difference in the density of eigenvalues on the two subspaces the bound would not show it up. Furthermore it is not applicable to the case $\mu < 1$ since if we try the obvious splitting

$$Z - G_\mu = Z - 2G_\mu^s - \tilde{G}_\mu \tag{3.10}$$

with $\tilde{G}_\mu$ as in (2.2) we now have G_μ^s positive definite on L^2_{even} and the problem becomes radically different. So we must try something else.

4. BOUNDS BASED ON THE BIRMAN–SCHWINGER PRINCIPLE

We use a slightly modified form of the Birman–Schwinger (BS) principle used to get bounds for the Schrödinger equation (Ref. 4, Section XIII.3C). To do this we first replace G_μ by some positive trace class operator L_μ such that $Z - G_\mu \geqslant Z - L_\mu$ on the subspace in question (this method does not work on L^2 as a whole). Then, considering the family of operators $Z - tL_\mu$, $t \in (0,1)$ for μ fixed we find that $n_{\mu,e}(\lambda)$, respectively, $n_{\mu,o}(\lambda)$, is exactly the number of $t \in (0,1)$ for which $Z - tL_\mu$ has an eigenvalue λ; this in turn is the number of eigenvalues in $(1,\infty)$ of the operator $(Z-\lambda)^{-1/2}L_\mu(Z-\lambda)^{-1/2}$ which in turn is bounded above by the trace of the latter operator (which by construction is positive).

We immediately get a first such result using as L_μ our H_μ^+ of the previous section. For $\lambda \in (0.1)$ we have, using the standard formula for the trace of continuous kernels,

$$\begin{aligned} n_\mu(\lambda) &\leqslant \mu^2\varphi_\alpha^+(0)\int_{-\infty}^{\infty} e^{-x^2}\left[Z(x)-\lambda\right]^{-1}dx \\ &= \mu^2\alpha^{-1/2}\varphi_1^+(0)\int_{-\infty}^{\infty} e^{-x^2}\left[Z(x)-\lambda\right]^{-1}dx \end{aligned} \tag{4.1}$$

The interesting aspect of this simple bound is that it goes like μ asymptotically—the same degree of divergence as conjectured on the basis of a nonrigorous treatment in Ref. 3. Formula (3.10) above shows us how to get an analogous result for $\mu < 1$ by decomposing $\tilde{G}_\mu$ in the same way as G_μ

before ($\tilde{G}_\mu = \tilde{H}_\mu^+ - \tilde{H}_\mu$ with $\tilde{H}_\mu^+ = \mu^2 e^{-2s^2x^2}\tilde{\Psi}_\mu^+(k)e^{-2s^2y^2}$) and taking now $L_\mu = 2G_\mu^s + \tilde{H}_\mu^+$.

We then argue as before but since we are now dealing with L^2_{even} only we must convert the operator $(Z-\lambda_0)^{-1/2}L_\mu(Z-\lambda_0)^{-1/2}$ into a kernel over $L^2(\mathbb{R}_+)$ before applying the trace formula. We get, setting from now on $\nu = \frac{1}{2} + \alpha$,

$$n_{\mu,e}(\lambda_0) \leqslant \mu^2\int_0^\infty \left[e^{-2\nu x^2}4x\sinh(2|\alpha|x^2) + e^{-4s^2x^2}\Psi_\mu^+(2x)\right]\frac{dx}{Z(x)-\lambda_0} \tag{4.2}$$

The first important thing to note about this formula is that if $Z(x)$ were growing somewhat faster (like $|x|^{2+\epsilon}$) the bound would converge to a finite value if $\mu\to\frac{1}{2}$; that is, in the limit we would only have finitely many discrete eigenvalues in any interval $(0,\lambda)$, $\lambda\in(0,1)$. The question whether this really is the case will be examined below from a different angle and turn out to be a tricky one. In the Rayleigh regime this is no problem (Ref. 1, Section 3) and this is reflected in the bound (4.1), which diverges regardless of the growth of $Z(x)$. To find the degree of divergence of the bound (4.2) we first note that the second summand in the integrand gives a finite contribution for all $\mu\in[\frac{1}{2},1]$ since $\Psi_\alpha^+(x)$ is the function $\varphi_1^+(|\alpha|x)$ as defined above apart from a constant that remains bounded. The latter function, however, is in $L^1(\mathbb{R}_+)$. The first summand we estimate as follows:

$$4\mu^2\left(\sup_{x\in\mathbb{R}_+}\frac{x}{Z-\lambda_0}\right)\int_0^\infty e^{-2\nu x^2}\sinh(2|\alpha|x^2)\,dx$$

$$= M_{\lambda_0}\mu^2\left[(1+4\alpha)^{-1}-1\right] = M_{\lambda_0}\frac{\mu^2}{2}s^{-1} \tag{4.3}$$

with a suitable constant M_{λ_0}. We summarize the results in the following:

Theorem 4.1. The total number of discrete eigenvalues of $Z - G_\mu$ in $[0,\lambda]$ is bounded by a multiple of the mass ratio γ for $\gamma\to\infty$ and by a multiple of γ^{-1} for $\gamma\to 0$.

The question now is: Is the difference in degree of divergence of $n_\mu(1)$ as given by the Schrödinger-bound above and the present $n_\mu(\lambda_0)$ real or an artifact of our method? Unfortunately in the bounds above we cannot let simply $\lambda_0\to 1$. This means that we have to construct our L_μ by a different method we want to estimate $n_\mu(1)$ by the BS method. Here is where we must consider the three subspaces (L^2_{even}, $\mu \gtrless 1$; L^2_{odd}, $\mu > 1$) separately. The easiest case to handle is the last one for which we shall carry out the necessary computations in some detail; for the other case we shall merely indicate the necessary adjustments and give only the results.

Let, therefore, $\mu > 1$ and f an odd, real function. A first possibility is to write $G_\mu = G_\mu^c + G_\mu^s$, to forget about the negative definite G_μ^c, and to extract a positive operator from the remainder as follows. The matrix element $(f, G_\mu^s f)$ may be evaluated term by term after expanding the sinh in the kernel.[1] The nth-order term in that expression reads

$$\iint_{-\infty}^{\infty} f(x)f(y)(xy)^{2n+1}|x-y|e^{-\nu(x^2+y^2)}\,dx\,dy$$

$$= 8\int_0^\infty dx\, f(x)x^{2n+2}e^{-\nu x^2}\int_0^x y^{2n+1}f(y)e^{-\nu y^2}\,dy$$

$$= 8\int_0^\infty dx\, f(x)x^{2n+2}e^{-\nu x^2}\int_0^\infty dy\, f(y)y^{2n+1}e^{-\nu y^2}$$

$$- 8\int_0 dx\, f(x)x^{2n+2}e^{-\nu x^2}\int_x^\infty dy\, f(y)y^{2n+1}e^{-\nu y^2} \tag{4.4}$$

Introducing the function

$$\chi_n(x) := \int_x^\infty y^{2n+1}e^{-\nu y^2}f(y)\,dy$$

the second term on the right-hand side of (4.4) simply becomes

$$-4\int_0^\infty \chi_n^2(x)\,dx < 0.$$

The first term we estimate as $ab \leqslant \frac{1}{2}(a^2 + b^2)$, i.e., by

$$2\left\{\left[\int_{-\infty}^\infty f(x)|x|x^{2n+1}e^{-\nu x^2}\,dx\right]^2 + \left[\int_{-\infty}^\infty f(x)x^{2n+1}e^{-\nu x^2}\,dx\right]^2\right\}$$

$$= 2\iint_{-\infty}^{\infty} f(x)f(y)(xy)^{2n+1}(1+|xy|)e^{-\nu(x^2+y^2)}\,dx\,dy \tag{4.5}$$

Doing this to all orders and leaving out all the negative contributions we get

$$(f, G_\mu^s f) \leqslant 2\mu^2\iint_{-\infty}^{\infty} f(x)f(y)e^{-\nu(x^2+y^2)}(1+|xy|)\sinh(2\alpha xy)\,dx\,dy \tag{4.6}$$

The new kernel is positive definite by construction and we take it as our L_μ. To check that L_μ is trace class we write $\sinh(2\alpha xy)$ as a sum of exponentials and apply the reasoning of the proof of Ref. 1, Theorem 1 to each of the kernels resulting from that. Now we get a finite limit when letting $\lambda_0 \to 1$ in the trace formula, we have

$$n_{\mu,o}(1) \leqslant 4\mu^2\alpha\int_0^\infty e^{-2\nu x^2}\frac{\sinh(2\alpha x^2)}{2\alpha x^2}\,\frac{x^2}{Z(x)-1}\,dx \tag{4.7}$$

In the other two cases our trouble stems from the zero-order term which needs special treatment. It represents a self-adjoint, degenerate operator of rank 2. We subtract it to get the truncated kernel G_μ^1; this kernel can be handled exactly as above, so for $Z - G^1_{\mu|L^2_{\text{even}}}$ we get a bound for its number of discrete eigenvalues $m_\mu(1)$ in $(-\infty, 1]$. We then have $n_\mu(1) \leqslant m_\mu(1) + 2$ because of the following simple lemma:

Lemma 4.2. Let the (bounded or unbounded) self-adjoint operator A have n discrete eigenvectors $\{\chi_i\}_1^n$ in $(-\infty, a]$, its continuous spectrum being contained in $[a, \infty)$. Let T be self-adjoint, degenerate of rank m. Then $A + T$ has at most $n + m$ discrete eigenvectors in $(-\infty, a]$.

This lemma is a special case of a much more general fact; the proof of this simple version is easy and can safely be omitted.

To get quick results for the even subspaces we forget about G_μ^s in the case $\mu > 1$ and let it stand as it is in the case $\mu < 1$. We then have the following theorem:

Theorem 4.3. With the previous method of handling G_μ the BS principle yields the bounds

$$n_{\mu,o}(1) \leqslant 2\mu^2 \int_0^\infty e^{-2\nu x^2}(1 + x^2)\sinh(2\alpha x^2)\,\frac{dx}{Z(x) - 1}, \quad \text{for } \mu > 1 \tag{4.8}$$

$$n_{\mu,e}(1) \leqslant 2\mu^2 \int_0^\infty e^{-2\nu x^2}(1 + x^2)\left[\cosh(2\alpha x^2) - 1\right]\frac{dx}{Z - 1} + 2, \quad \text{for } \mu > 1 \tag{4.9}$$

$$\begin{aligned} n_{\mu,e}(1) \leqslant\ & 2\mu^2 \int_0^\infty e^{-2\nu x^2}(1 + x^2)\left[\cosh(2|\alpha| x^2) - 1\right]\frac{dx}{Z - 1} \\ & + 2\mu^2 \int_0^\infty e^{-2\nu x^2} 2x \sinh(2|\alpha| x^2)\,\frac{dx}{Z - 1} + 2, \quad \text{for } \mu < 1 \end{aligned} \tag{4.10}$$

In particular we get an explicit proof that on all three subspaces the total number of discrete eigenvalues is finite.

In order to find the degree of divergence as a function of μ of these expressions we state without proof [its methods are similar to those used in the proof of Lemma 1 in Ref. 1 or in the bounds of $L(x)$ in Section 1] the following lemma:

Lemma 4.4. For all $x \in \mathbb{R}$ we have the double inequality

$$\frac{1}{4}(2 + x) \leqslant x^2\left[Z(x) - 1\right]^{-1} \leqslant 1 + \frac{x}{\sqrt{\pi}} \tag{4.11}$$

From this it follows immediately that by computing an upper bound on the integrals in Theorem 4.3 by means of the second inequality of the lemma we will get the correct degree of divergence. We carry out this procedure now for the first case, the other ones are treated similarly. After a change of variables we have to compute ($t := \alpha/\nu$)

$$\frac{1}{(2\nu)^{1/2}}\int_0^\infty e^{-x^2}\left\{\sinh(tx^2)\left[1+\frac{x}{(2\pi\nu)^{1/2}}\right] + 2\alpha\left[1+\frac{x}{(2\pi\nu)^{1/2}}\right]\frac{\sinh(tx^2)}{tx^2}\right\}dx \tag{4.12}$$

Putting in the power series of $\sinh(tx^2)$ and integrating termwise (this is allowed here) we get

$$\frac{1}{(2\nu)^{1/2}}\left\{\frac{1}{4}\left[\left(\frac{\pi\nu}{\nu-\alpha}\right)^{1/2}-\left(\frac{\pi\nu}{\nu+\alpha}\right)^{1/2}\right]+\frac{1}{2(2\pi\nu)^{1/2}}\left(\sum_{n=0}^{\infty}t^{2n+1}\right) + \alpha\left[\sum_0^\infty t^{2n}\frac{\Gamma(2n+1/2)}{(2n+1)!}\right]+\frac{\alpha}{(2\pi\nu)^{1/2}}\left[\sum_0^\infty\frac{t^{2n}}{(2n+1)}\right]\right\} \tag{4.13}$$

The first and third sum in (4.13) make no trouble and sum to [$t \in (0,1)$] $t(1-t^2)^{-1}$ and $(1/2t)\ln[(1+t)/(1-t)]$, respectively. For the coefficients in the second sum we use Stirling's formula to estimate $\Gamma(2n+1/2)[(2n+1)!]^{-1} \leqslant \sqrt{\pi}\,(2n+1)^{-3/2}$ so that the contribution of this sum does not diverge when $\mu \to \infty$ (i.e., $t \to 1$). In all we get

$$n_{\mu,o}(1) \leqslant 2\mu^2\left\{\frac{\sqrt{\pi}}{4}\left(1-\frac{1}{2\mu-1}\right)+\frac{1}{4\sqrt{\pi}}\,\frac{\alpha}{\alpha+1/4}+\frac{1}{2\sqrt{\pi}}\ln(2\mu-1) + \frac{\alpha\sqrt{\pi}}{(2\nu)^{1/2}}\left[\sum_0^\infty t^{2n}(2n+1)^{-3/2}\right]\right\} \tag{4.14}$$

This bound, therefore, diverges as μ^3, worse than the Schrödinger bound. On the other hand, it converges to zero for $\mu \to 1$, which the Schrödinger bound—even restricted to L^2_{odd}—does not. So this method has some merits. The result for (4.9) is the same: divergence like μ^3. The mode of divergence of (4.10) for $s \to 0$ is different, the methods above work nevertheless to give a divergence like s^{-2}.

It now turns out that we can do somewhat better with this method by handling the terms in the expansion of G_μ in powers of α in a more sophisticated manner. Let us therefore go back to the case $\mu > 1$, L^2_{odd} and introduce for each μ the functions

$$F^{\mu}_{2n}(x) := \int_{-\infty}^{x} y^{2n} e^{-\nu y^2} f(y)\,dy, \qquad G^{\mu}_{2n}(x) := \int_{-\infty}^{x} y F^{\mu}_{2n}(y)\,dy \quad (4.15)$$

for some fixed $f \in L^2_{\mathrm{odd}}$. Then clearly both functions F^{μ}_{2n} and G^{μ}_{2n} are in L^2_{even} for all $n \in \mathbb{N}$. Again we expand G^s_{μ} in powers of α and consider the term of order $2k+1$ of the matrix element $(f, G^s_{\mu} f)$:

$$\begin{aligned}
&\iint_{-\infty}^{\infty} f(x)f(y)|x-y|(xy)^{2k+1} e^{-\nu(x^2+y^2)}\,dx\,dy \\
&\quad = \iint_{-\infty}^{\infty} F^{\mu}_{2k}(x)(\partial_x \partial_y xy|x-y|) F^{\mu}_{2k}(y)\,dx\,dy \\
&\quad = 2\iint_{-\infty}^{\infty} F^{\mu}_{2k}(x) F^{\mu}_{2k}(y)|x-y|\,dx\,dy \\
&\quad = 16\int_0^{\infty} x F^{\mu}_{2k}(x)\,dx \int_0^x F_{2k}(y)\,dy = -8(F^{\mu}_{2k}, G^{\mu}_{2k})
\end{aligned} \quad (4.16)$$

On the other hand, we have for the term of order $2k$, $k \geqslant 1$, the two expressions

$$\begin{aligned}
-2\|F^{\mu}_{2k}\|_2^2 &= \iint_{-\infty}^{\infty} f(x)f(y)|x-y|(xy)^{2k} e^{-\nu(x^2+y^2)}\,dx\,dy \\
&= \iint_{-\infty}^{\infty} F^{\mu}_{2k-2}(x) F^{\mu}_{2k-2}(y)\left[\partial_x \partial_y (xy)^2 |x-y|\right] dx\,dy \\
&= -12\|G^{\mu}_{2k-2}\|_2^2
\end{aligned} \quad (4.17)$$

Therefore we can balance the terms of order $2k$ and $2k+1$ in the expansion of G_{μ} itself; we have, using (4.16) and (4.17),

$$\begin{aligned}
&-2\frac{(2\alpha)^{2k}}{(2k)!}\|F_{2k}\|^2 - 8\frac{(2\alpha)^{2k+1}}{(2k+1)!}(G^{\mu}_{2k}, F^{\mu}_{2k}) \\
&\quad = -2\frac{(2\alpha)^{2k}}{(2k)!}\left\|F_{2k} + \frac{4\alpha}{2k+1}G_{2k}\right\|_2^2 + \frac{4}{3}\,\frac{(2\alpha)^{2k+2}}{(2k+1)!\,(2k+1)}\|F_{2k+2}\|_2^2
\end{aligned}$$

(4.18)

Retaining only the second summand in (4.18) we get another—and we hope—subtler bound for $G_{\mu\,|\,L^2_{\text{odd}}}$

$$(f, G_\mu f) \leqslant -\frac{4}{3}\,\mu^2 \int\int_{-\infty}^{\infty} f(x) f(y)|x-y| e^{-\nu(x^2+y^2)} \times \left[\sum_{n=1}^{\infty} \frac{(2\alpha xy)^{2n}}{(2n-1)!\,(2n-1)} \right] dx\, dy \tag{4.19}$$

If we gather the neglected terms we get, incidentally, another decomposition of G_μ as a difference of two positive operators.

On L^2_{even} for $\mu > 1$ we first have to handle the zero-order term as in the previous bound; for the truncated operator $G^1_{\mu\,|\,L^2_{\text{even}}}$ we get with the same reasoning as above:

$$(f, G_\mu f) \leqslant -\frac{4}{3}\,\mu^2 \int\int_{-\infty}^{\infty} f(x) f(y)|x-y| e^{-\nu(x^2+y^2)} \times \left[\sum_{n=1}^{\infty} \frac{(2\alpha xy)^{2n+1}}{(2n)!\,(2n)} \right] dx\, dy \tag{4.20}$$

In the Lorentz regime there is, of course, no balancing of terms in G^c_μ and G^s_μ, instead we write for $k > 1$

$$\left| 8\,\frac{(2\alpha)^{2k}}{(2k)!}\,(G^\mu_{2k-1}, F^\mu_{2k-1}) \right| \leqslant 2\,\frac{(2\alpha)^{2k-1}}{(2k-1)!} \left(\|F^\mu_{2k-1}\|^2 + \left\| \frac{4\alpha}{2k}\, G^\mu_{2k-1} \right\|^2 \right)$$

and add the resulting contributions to G^s_μ using (4.17) again. This gives

$$(f, G_\mu f) \leqslant 2(f, G^s_\mu f) + \frac{4}{3}\,\mu^2 \int\int_{-\infty}^{\infty} f(x) f(y)|x-y| e^{-\nu(x^2+y^2)} \times \left[\sum_{n=1}^{\infty} \frac{(2\alpha xy)^{2n+1}}{(2n)!\,(2n)} \right] dx\, dy \tag{4.21}$$

This is a little more complicated but for estimating the degree of divergence of $n_{\mu,e}(1)$ from the bound (4.21) ($\mu < 1$) it is readily seen that we loose nothing by estimating the second summand on the right of (4.21) by $2(f, G^s_\mu f)$—it makes no difference.

If we use the bounds (4.19), (4.20), and (4.21) to apply the BS principle we find that they give the same answer as the Schrödinger bound for $n_\mu(\lambda_0)$, $\lambda_0 \in (0.1)$. However, for $n_{\mu,e}(1)$, respectively, $n_{\mu,0}(1)$, the new method gives considerably better results; for the Rayleigh regime we find that $n_{\mu,e}(1)$ and $n_{\mu,o}(1)$ both diverge as $\mu^2 \ln(2\mu - 1)$—better than before but still worse that the Schrödinger bound. This seems surprising if we recall how drastically we cut down $Z(x)$ in that case. In the Lorentz regime, however, we find

$$n_{\mu,e}(1) \leqslant 2 + 8\mu^2 t \int_0^\infty x e^{-x^2} \left[1 + \frac{x}{(2\pi\nu)^{1/2}} \right] \frac{\sinh(tx^2)}{tx^2}\, dx \qquad (4.22)$$

with $t := |\alpha|/\nu$. One contribution to the integral gives a logarithmic divergence; the other one sums to $\sum_{n=2}^{\infty} t^{2n}\Gamma(2n+3/2)(2n+1)!)^{-1}$. This expression also occurs in the previous bounds but now it is the leading term and thus needs more careful handling. Stirling's formula again shows that apart from lower order terms it goes like $\sum_{n=1}^{\infty} t^{2n}(2n)^{-1/2}$. This can be converted with the usual trick into an integral over $[1, \infty)$; integrating that by parts we find that it diverges as $(\ln t)^{-1}$, i.e., it goes like s^{-1}. So at least for the Lorentz regime the extra trouble has paid off and the conjecture mentioned above has been confirmed. We summarize the results obtained with the Birman–Schwinger method in

Theorem 4.3. For $n_\mu(\lambda)$, $\lambda \in (0.1]$ we have

(a) $n_\mu(\lambda) \leqslant c_1\mu$, respectively, $n_\mu(\lambda) \leqslant c_2 s^{-1}$ for $\lambda \in (0.1)$

(b) $n_\mu(1) \leqslant c_3\mu^2 \ln(2\mu - 1)$ for $\mu > 1$

(c) $n_{\mu,e}(1) \leqslant c_4 s^{-1}$ for $\mu < 1$

As yet we cannot prove that the degree of divergence of $n_\mu(1)$ in the two regimes actually is different though there are indication that this is the case. So far we do not even know that $n_{\mu,e}(1)$ actually diverges as $\mu \to 1/2$. Why is that so difficult?

In Ref. 1 we showed that the symmetric operator $Z - G_{1/2}$ with domain $D(Z)$ is essentially self-adjoint; $Z - G_\mu$ converges to it strongly in the generalized sense. From Theorem VIII-1.14 in Ref. 5 we conclude that we must show that the essential spectrum of $Z - G_{1/2}$ is the whole of $[0, \infty)$. Viewing it as an operator on $L^2(\mathbb{R}_+)$ and applying Weyl's criterion on invariance of the essential spectrum with respect to compact perturbations to its square we wind up with a perturbed Sturm–Liouville operator to whose perturbation none of the current theorems can be applied; so this question is a highly nontrivial one.

Before closing let us mention one point which the reader may already have wondered about. We could have estimated closer, in the bound (4.1), say, by not throwing away H_μ^- as we did; that would have yielded

$$n_\mu(\lambda) \leqslant \mathrm{Tr}\big((Z + H_\mu^- - \lambda)^{-1} H_\mu^+\big), \qquad \lambda \in (0.1) \qquad (4.23)$$

Similar possibilities exist in all the cases where G_μ has been decomposed as difference of two positive operators. The reason why we did not do this is that the inverse of $Z + H_\mu^- - \lambda$ cannot be computed except as a series expansion with respect to the perturbation H_μ^-. The first order of this perturbation expansion gives a contribution (the *ad hoc* introduction of functions F and θ should be self-explanatory)

$$
\begin{aligned}
-\iint_{-\infty}^{\infty} & [Z(x)-\lambda]^{-1} H_\mu^-(x,y)[Z(y)-\lambda]^{-1} H^+(x,y)\,dx\,dy \\
&= -\mu^4 \iint_{-\infty}^{\infty} \frac{e^{-x^2}e^{-y^2}}{[Z(x)-\lambda][Z(y)-\lambda]} \varphi_\alpha^+(x-y)\varphi_\alpha^-(x-y)\,dx\,dy \\
&= -\frac{\mu^4}{\alpha} \iint F(x)F(y)\theta\left[\sqrt{\alpha}\,(x-y)\right]dx\,dy \\
&= -\frac{\mu^4}{\alpha^{3/2}} \int_{-\infty}^{\infty} |\tilde{F}(k)|^2 \tilde{\theta}\left(\frac{k}{\sqrt{\alpha}}\right)\frac{dk}{\sqrt{2\pi}}
\end{aligned}
\tag{4.24}
$$

and this diverges far slower with $\mu \to \infty$ than the leading term. The same holds true for the higher-order terms, i.e., the asymptotic behavior does not change if the more complicated bound is used. Only for small μ is there a sizable improvement on the bound (4.1) but there the original bound is extremely weak, so that does not seem worth the trouble. Though one could possibly improve the bounds we gave along the same lines of reasoning we feel that radically different methods will have to be used if a substantial improvement is to be gotten, supposing, of course, that our bounds do not yet give the correct asymptotic behavior.

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REFERENCES

1. W. Driessler, On the spectrum of the Rayleigh piston, *J. Stat. Phys.* **24**:595–606 (1981).
2. B. Simon, An introduction to the self-adjointness and spectral analysis of Schrödinger operators, in *Acta Phys. Austr. Suppl.* **XVII**:19–42 (1977).
3. M. R. Hoare, and R. Rahman, On the spectral theory of Rayleigh's piston: I. The discrete spectrum, *J. Phys. A* **6**:1461–1478 (1973).
4. M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, New York (1978).
5. T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin (1966).